

Negative energy ground states for the L^2 -critical NLSE on metric graphs

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Abstract

We investigate the existence of ground states with prescribed mass for the focusing nonlinear Schrödinger equation with L^2 -critical power nonlinearity on noncompact quantum graphs. We prove that, unlike the case of the real line, for certain classes of graphs there exist ground states with negative energy for a whole interval of masses. A key role is played by a thorough analysis of Gagliardo-Nirenberg inequalities and on estimates of the optimal constants. Most of the techniques are new and suited to the investigation of variational problems on metric graphs.

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1 Introduction

In this paper we investigate the existence of ground states for the *critical* NLS energy functional

$$(1) \quad E(u, \mathcal{G}) = \frac{1}{2} \|u'\|_{L^2(\mathcal{G})}^2 - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6 = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \frac{1}{6} \int_{\mathcal{G}} |u|^6 dx$$

on a noncompact metric graph \mathcal{G} , under the *mass constraint*

$$(2) \quad \|u\|_{L^2(\mathcal{G})}^2 = \mu.$$

The subcritical case, where the L^6 norm is replaced by an L^p norm with $p \in (2, 6)$, has been investigated in [6, 7]. The energy in (1) is *critical* in the sense that, under the mass-preserving transformations

$$u(x) \mapsto u_{\lambda}(x) := \lambda^{1/2} u(\lambda x) \quad (\lambda > 0),$$

the kinetic and the potential terms in (1) scale in the same way, namely

$$(3) \quad E(u_{\lambda}, \lambda^{-1} \mathcal{G}) = \lambda^2 E(u, \mathcal{G}),$$

which is typical of critical problems with a strong loss of compactness.

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Throughout the paper, \mathcal{G} denotes a *noncompact metric graph*, i.e. a connected metric space obtained by gluing together, by the identification of some of their endpoints, a finite number of closed line intervals (not necessarily bounded), according to the topology of a graph, self-loops and multiple edges being allowed. Any bounded edge e is identified with an interval $[0, \ell_e]$, while unbounded edges are referred to as “half-lines”, and are identified with (copies of) the positive half-line $\mathbb{R}^+ = [0, +\infty)$; at least one edge is assumed to be unbounded, so that \mathcal{G} is noncompact (two very special cases are when $\mathcal{G} = \mathbb{R}^+$ and when $\mathcal{G} = \mathbb{R}$, the latter being obtained by gluing together two copies of \mathbb{R}^+). We refer to Section 2 (see also [11, 23, 6]) for more details.

In this framework, by a “ground state of mass μ ” we mean a solution to the minimization problem

$$(4) \quad \min_{u \in H_\mu^1(\mathcal{G})} E(u, \mathcal{G}), \quad H_\mu^1(\mathcal{G}) := \left\{ u \in H^1(\mathcal{G}) : \|u\|_{L^2(\mathcal{G})}^2 = \mu \right\},$$

for which it is clearly sufficient to work with real valued, nonnegative functions. Obviously ground states solve, for some $\omega \in \mathbb{R}$, the stationary quintic NLS equation

$$u'' + |u|^4 u = \omega u$$

on each edge of \mathcal{G} , with Kirchhoff boundary conditions at the vertices (see Prop. 3.3 in [6]).

The existence of ground states for a given μ is strictly related to the behavior of the *ground-state energy level function*

$$(5) \quad \mathcal{E}_{\mathcal{G}}(\mu) = \inf_{u \in H_\mu^1(\mathcal{G})} E(u, \mathcal{G}), \quad \mu \geq 0,$$

which will play a central role throughout this paper.

As is wellknown (see Sec. 2), when $\mathcal{G} = \mathbb{R}$ there exists a *critical mass* $\mu_{\mathbb{R}}$ such that the minimization problem (4) has a solution if and only if $\mu = \mu_{\mathbb{R}}$, and the same occurs when $\mathcal{G} = \mathbb{R}^+$ (with a *smaller* critical mass $\mu_{\mathbb{R}^+} = \mu_{\mathbb{R}}/2$). This severe restriction is due to the scaling rule (3) and the dilation-invariance of \mathbb{R} and \mathbb{R}^+ . Thus, when $\mathcal{G} = \mathbb{R}$ or $\mathcal{G} = \mathbb{R}^+$, the minimization process (4) is extremely unstable and, in a sense, of little interest.

When \mathcal{G} is a generic (noncompact) metric graph, however, the problem can be highly nontrivial and, depending on the topology of \mathcal{G} , entirely new phenomena may arise, such as problem (4) having solutions if, and only if, μ belongs to some *whole interval* of masses.

For each graph \mathcal{G} we can define, in a natural way, a *critical mass* $\mu_{\mathcal{G}}$, that depends on \mathcal{G} via the best constant $K_{\mathcal{G}}$ in the Gagliardo-Nirenberg inequality (10), and it turns out that $\mu_{\mathbb{R}^+} \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}$, so that \mathbb{R}^+ and \mathbb{R} are extremal graphs, as concerns the critical mass (see Proposition 2.3). The mass $\mu_{\mathcal{G}}$ is the precise threshold such that $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ (possibly $-\infty$) as soon as $\mu > \mu_{\mathcal{G}}$ and, on a general ground, a *necessary condition* for the existence of ground states in (4) is that $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$ (see Proposition 2.4).

This condition, however, is far from being sufficient: the true nature of problem (4) strongly depends on the topology of \mathcal{G} , and the following (mutually exclusive) cases are possible:

- (a) \mathcal{G} has a *terminal point* (a tip, Fig. 1). Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$, and problem (4) has no solution unless $\mu = \mu_{\mathbb{R}^+}$ and \mathcal{G} is isometric to \mathbb{R}^+ .
- (b) \mathcal{G} admits a *cycle covering* (Fig. 2). Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$, and problem (4) has no solution unless $\mu = \mu_{\mathbb{R}}$ and \mathcal{G} is isometric to \mathbb{R} (or to one of a few very special, and completely classified, other structures, see Theorem 2.5 in [6]).

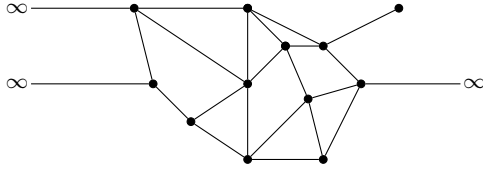


Figure 1: case (a).

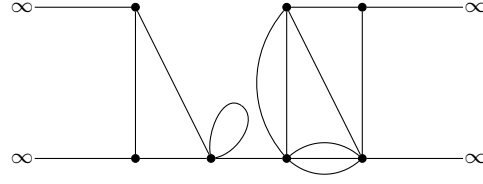


Figure 2: case (b).

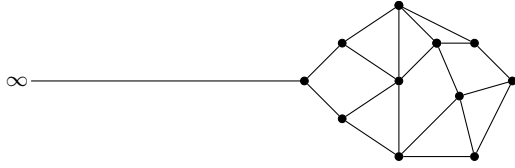


Figure 3: case (c).

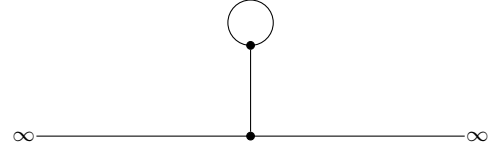


Figure 4: case (d).

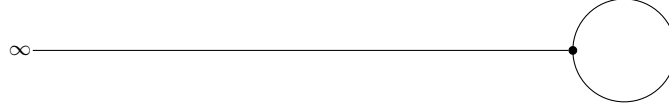


Figure 5: a tadpole graph.

- (c) \mathcal{G} has exactly one half-line and no terminal point (Fig. 3). Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$, and problem (4) has a solution if and only if $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$.
- (d) In all other cases (Fig. 4): if $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$, then problem (4) has a solution if and only if $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$.

Some remarks are in order, to better clarify the scope of this scenario (the precise statements, which are the main results of the paper, are given in Theorems 3.1, 3.2, 3.3 and 3.4).

In the first two cases ground states, as a rule, do not exist. In case (a), the presence of a tip—hence of a terminal edge—allows the construction of “monotone” functions of mass $\mu_{\mathbb{R}^+}$, that decrease away from the tip and mimic a half-soliton on \mathbb{R}^+ , with an energy level arbitrarily close to zero (albeit strictly positive, unless \mathcal{G} is exactly \mathbb{R}^+): thus, in a sense, graphs with a tip behave much like a half-line. In case (b), by contrast, the covering assumption is not compatible with “monotone” functions and, due to a rearrangement argument from \mathcal{G} to \mathbb{R} , no function of mass $\mu_{\mathbb{R}}$ can have a negative energy on \mathcal{G} . This rigidity rules out ground states, unless \mathcal{G} supports a soliton, and this, in turn, occurs only when $\mathcal{G} = \mathbb{R}$ (possibly with the identification of some pairs of points, in a way compatible with the even symmetry of a soliton). Thus, dually, a graph as in (b) behaves much like \mathbb{R} .

The last two cases are, on the contrary, extremely nontrivial. In (c), \mathcal{G} consists of a compact core \mathcal{K} (with no terminal edge) attached to a half-line, the simplest example being the “tadpole” graph in Fig. 5.

If $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, a ground state of mass μ always exists—with a *strictly negative* energy—and this is a completely new phenomenon. Over \mathbb{R}^+ , due to (3), one has $\mathcal{E}_{\mathbb{R}^+}(\mu) = -\infty$

(and no ground state) as soon as $\mu > \mu_{\mathbb{R}^+}$: here, on the contrary, the compact core \mathcal{K} attached to \mathbb{R}^+ has the effect of a *stabilizer* as regards ground states: due to \mathcal{K} , \mathcal{G} loses dilation invariance, and high concentration is no longer energetically convenient, which accounts for *strictly negative* (yet finite!) ground-state energy levels.

Finally, in (d), \mathcal{G} has no tip, no cycle covering and (being noncompact) at least two half-lines. This case becomes very interesting if one *further* assumes that $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$, which guarantees the existence of ground states for every mass $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$: here, contrary to (c), a ground state exists also when $\mu = \mu_{\mathcal{G}}$, with a zero energy level. A particularly interesting, and specific, feature of the case $\mu = \mu_{\mathcal{G}}$ is the coexistence of compact and noncompact minimizing sequences. Thus an additional difficulty in this case is the *choice* of a proper sequence to work with.

Explicit examples of graphs can be constructed (e.g. the “signpost” graph in Fig. 4, as explained in Sec. 3), where $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$: this extra assumption, however, is crucial to prove the existence of ground states, and we believe that it cannot be dropped in general.

More precisely we believe that, within case (d), the *sole* topology of \mathcal{G} is not enough, in general, to establish whether $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$ or $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$ and, in the latter case, whether a ground states exists, of mass $\mu_{\mathcal{G}}$. It is an open problem, at present, to fully understand problem (4) in case (d), when $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$.

Thus, summing up, the four cases (a)–(d) cover all the possible topologies of a (non-compact) metric graph \mathcal{G} . The first two cases are extremely rigid, with ground states being the exception rather than the rule. Case (c) is, on the contrary, very interesting, with ground states in a universal range of prescribed masses, and one may consider such graphs as “intermediate” between \mathbb{R}^+ and \mathbb{R} . Finally, case (d) is also nontrivial (with ground states in a whole, closed interval of prescribed masses), but this is subordinated to the condition that $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$: in this case \mathcal{G} shows, again, an intermediate behavior between \mathbb{R}^+ and \mathbb{R} . To conclude this discussion we wish to emphasize that all the ground states of cases (c) and (d) (except those of mass $\mu_{\mathcal{G}}$) have *negative energy*. This is in sharp contrast with the behavior of the NLS equation on \mathbb{R} (or \mathbb{R}^n), where every solution with negative energy blows up in finite time ([16]), and will be the object of a forthcoming paper.

The problem of the minimization of (1) under the constraint (2) can be interpreted according to the Gross-Pitaevskii theory for the ground state of the Bose-Einstein condensates. Indeed, following a series of results obtained in the last decade by several authors (see e.g. [26, 27, 28, 5, 20, 21, 22, 33, 10, 18]), under some physical conditions the dynamics of a gas of interacting identical bosons can be described through a one-body nonlinear equation, called *Gross-Pitaevskii equation*. More precisely, if the particles in the gas interact in pairs, then the resulting equation displays a *cubic* nonlinearity. On the other hand, in [17] a system of identical bosons interacting through a *three-body* potential was considered, and the resulting equation was shown to be a *quintic* NLS.

In actual dilute Bose-Einstein condensates, two- and three-body interactions may co-exist, so that in the resulting one-body equation both cubic and quintic terms would arise, even though normally the effect of the cubic term overwhelms the effect of the quintic, that is therefore neglected. Concerning the sign of the nonlinear terms, it turns out to depend on the character attractive or repulsive of the interaction among the particles, so that it is possible to realize experimentally condensates that display either a focusing or a defocusing behaviour ([19]).

For these reasons, and since both the functional (1) and the L^2 -norm are conserved by the evolution driven by the quintic NLS equation

$$(6) \quad i\partial_t u(t, x) = -u''(t, x) - |u(t, x)|^4 u(t, x),$$

the issue of the existence of a minimizer of the constrained energy can be interpreted as the search for the ground state of a particular Bose-Einstein condensate where two-body interactions are absent.

With respect to the problems currently studied as regards ground states of a Bose-Einstein condensate and to the rigorous derivation of equation (6) obtained in [17], let us stress two remarkable differences: first, we consider a *focusing* nonlinearity, second, we set the problem on a graph. Both features have nowadays an established experimental counterpart: on the one hand, self-concentrating condensates are currently realised [19], on the other hand, condensation on graph-like structures has been recently observed in [29].

While linear dynamics on quantum graphs is nowadays a well-known branch of mathematical physics (see e.g. [11, 23, 34]), its nonlinear counterpart has gained an increasing interest only quite recently. A seminal study of nonlinear evolution equation on ramified structures appeared in [8] and then was extended to several physical domains, involving various mathematical issues [12, 9]. The study of the evolution of solitary waves on star graphs was performed in [1], while the search for ground states was carried out in [36, 35, 2, 3, 4] for the case of star graphs, and then extended to more general graphs (dealing with the same class considered in this paper) in [6, 7, 25]. Stationary states and related bifurcation were recently investigated in [14, 30, 31]. In particular, [32] treats the case of periodic graphs, which is not covered in this paper, and shows the occurrence of a bifurcation phenomenon. The integrability of the cubic NLS on star graphs was proved in [15]. All cited papers deal with the subcritical case or even specialize to the cubic case. To our knowledge, the present work is the first contribution to the understanding of the role of the L^2 -criticality in the framework of graphs. The fact that this role appears to be different from what happens on standard domains like \mathbb{R}^n is in our opinion worth being stressed.

The paper is organised as follows: in Section 2 we give some preliminary results and introduce the notion of critical mass; in Section 3 we state the four main theorems. The core of the proofs is a result stated and proved in Section 4. Finally, in Section 5 we conclude the proofs of the main existence results stated in Section 3.

2 Notation and preliminary results

It is well known ([16]) that when $\mathcal{G} = \mathbb{R}$, the ground-state energy level function, as defined in (5), has a sharp transition from 0 to $-\infty$, corresponding to a special value $\mu_{\mathbb{R}}$ of the mass, known as the *critical mass*:

$$(7) \quad \mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_{\mathbb{R}} \\ -\infty & \text{if } \mu > \mu_{\mathbb{R}} \end{cases} \quad \left(\mu_{\mathbb{R}} = \pi\sqrt{3}/2 \right).$$

Furthermore, the infimum $\mathcal{E}_{\mathbb{R}}(\mu)$ is attained (i.e. a ground state exists in (4)) *if and only if* $\mu = \mu_{\mathbb{R}}$. Thus every ground state u (necessarily of mass $\mu_{\mathbb{R}}$) satisfies $E(u, \mathbb{R}) = 0$. The ground states, called *solitons*, form a quite large family: up to phase multiplication and translations, they can be written as

$$(8) \quad \phi_{\lambda}(x) = \sqrt{\lambda}\phi(\lambda x), \quad \lambda > 0,$$

where

$$(9) \quad \phi(x) = \operatorname{sech}^{1/2} \left(2x/\sqrt{3} \right).$$

When $\mathcal{G} = \mathbb{R}^+$ (the positive half-line), the situation is similar, but with the proper critical mass $\mu_{\mathbb{R}^+} = \mu_{\mathbb{R}}/2$:

$$\mathcal{E}_{\mathbb{R}^+}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_{\mathbb{R}^+} \\ -\infty & \text{if } \mu > \mu_{\mathbb{R}^+} \end{cases} \quad \left(\mu_{\mathbb{R}^+} = \pi\sqrt{3}/4 \right).$$

Again, ground states of mass μ exist if and only if $\mu = \mu_{\mathbb{R}^+}$. They are called “half-solitons”, as they are the restrictions to \mathbb{R}^+ of the family ϕ_λ .

Thus, when \mathcal{G} is \mathbb{R} or \mathbb{R}^+ , problem (4) is trivialized by these sharp transitions.

For a general noncompact graph \mathcal{G} , the behavior of $\mathcal{E}_{\mathcal{G}}(\mu)$ is strictly related to the Gagliardo–Nirenberg inequality

$$(10) \quad \|u\|_{L^6(\mathcal{G})}^6 \leq K_{\mathcal{G}} \|u\|_{L^2(\mathcal{G})}^4 \|u'\|_{L^2(\mathcal{G})}^2, \quad \forall u \in H^1(\mathcal{G})$$

valid for every noncompact graph (see [7, 37]). The number $K_{\mathcal{G}}$ is the *best constant* that one can put in (10), namely,

$$(11) \quad K_{\mathcal{G}} = \sup_{\substack{u \in H^1(\mathcal{G}) \\ u \neq 0}} \frac{\|u\|_{L^6(\mathcal{G})}^6}{\|u\|_{L^2(\mathcal{G})}^4 \cdot \|u'\|_{L^2(\mathcal{G})}^2} = \sup_{u \in H_{\mu}^1(\mathcal{G})} \frac{\|u\|_{L^6(\mathcal{G})}^6}{\mu^2 \cdot \|u'\|_{L^2(\mathcal{G})}^2},$$

where the last equality follows from homogeneity.

The role of this constant, in connection with the behavior of $\mathcal{E}_{\mathcal{G}}(\mu)$, is clear: recalling (1) and the definition of $H_{\mu}^1(\mathcal{G})$ in (4), using (10) we have, for every $u \in H_{\mu}^1(\mathcal{G})$,

$$E(u, \mathcal{G}) \geq \frac{1}{2} \|u'\|_{L^2(\mathcal{G})}^2 - \frac{1}{6} K_{\mathcal{G}} \mu^2 \|u'\|_{L^2(\mathcal{G})}^2 = \frac{1}{6} \|u'\|_{L^2(\mathcal{G})}^2 (3 - K_{\mathcal{G}} \mu^2).$$

We then see that

$$(12) \quad \mu^2 \leq 3/K_{\mathcal{G}} \implies E(u, \mathcal{G}) \geq 0 \quad \text{for all } u \in H_{\mu}^1(\mathcal{G}).$$

Note also that

$$(13) \quad \mu^2 < 3/K_{\mathcal{G}} \implies E(u, \mathcal{G}) > 0 \quad \text{for all } u \in H_{\mu}^1(\mathcal{G}).$$

On the other hand, if $\mu^2 > 3/K_{\mathcal{G}}$, say $\mu^2 = 3(1 + \delta)/K_{\mathcal{G}}$ for some $\delta > 0$, we can take $u \in H_{\mu}^1(\mathcal{G})$ close to optimality in (11), say

$$\|u\|_{L^6(\mathcal{G})}^6 > \frac{K_{\mathcal{G}}}{1 + \delta} \mu^2 \|u'\|_{L^2(\mathcal{G})}^2,$$

to obtain

$$(14) \quad E(u, \mathcal{G}) < \frac{1}{6} \|u'\|_{L^2(\mathcal{G})}^2 \left(3 - \frac{K_{\mathcal{G}}}{1 + \delta} \mu^2 \right) = 0.$$

This shows that

$$(15) \quad \mu^2 > 3/K_{\mathcal{G}} \implies E(u, \mathcal{G}) < 0 \quad \text{for some } u \in H_{\mu}^1(\mathcal{G}).$$

Now (15) and (12) justify the following definition.

Definition 2.1. The *critical mass* for a noncompact metric graph \mathcal{G} is the number

$$\mu_{\mathcal{G}} = \sqrt{3/K_{\mathcal{G}}}.$$

This definition, of course, gives the correct critical mass when \mathcal{G} is \mathbb{R} or \mathbb{R}^+ . In general, from (13) and (15), we see that $\mu_{\mathcal{G}}$ is the precise mass threshold, after which the ground-state energy level $\mathcal{E}_{\mathcal{G}}(\mu)$ becomes negative (possibly $-\infty$).

Remark 2.2. Any noncompact metric graph \mathcal{G} has (at least) one unbounded edge which, in turn, contains arbitrarily large intervals. Therefore, any function $v \in H^1(\mathbb{R})$ having *compact support* can be regarded as an element of $H^1(\mathcal{G})$, by placing the support of v inside a half-line of \mathcal{G} , and setting $v \equiv 0$ outside. Thus, in a sense, $H^1(\mathcal{G})$ “contains” a dense subset of $H^1(\mathbb{R})$.

Next we notice that any noncompact \mathcal{G} is, in a way, intermediate between \mathbb{R}^+ and \mathbb{R} in the sense of the following statement.

Proposition 2.3. *Let \mathcal{G} be a noncompact graph, and let $\mu_{\mathcal{G}}$ be the critical mass for \mathcal{G} . Then*

$$(16) \quad \mu_{\mathbb{R}^+} \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}$$

or, equivalently,

$$(17) \quad K_{\mathbb{R}} \leq K_{\mathcal{G}} \leq K_{\mathbb{R}^+}.$$

Moreover, we also have

$$(18) \quad \mathcal{E}_{\mathbb{R}^+}(\mu) \leq \mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathbb{R}}(\mu), \quad \forall \mu > 0.$$

Proof. Given $u \in H^1(\mathcal{G})$ (assume $u \geq 0$ and $u \not\equiv 0$), let $u^* \in H^1(\mathbb{R}^+)$ be its decreasing rearrangement on \mathbb{R}^+ (for properties of rearrangements on graphs see [6, 24]). Since

$$\|(u^*)'\|_{L^2(\mathbb{R}^+)}^2 \leq \|u'\|_{L^2(\mathcal{G})}^2, \quad \|u^*\|_{L^p(\mathbb{R}^+)}^p = \|u\|_{L^p(\mathcal{G})}^p \quad \forall p,$$

the quotient for u in (11) does not exceed the same quotient for u^* (over \mathbb{R}^+): since the latter is bounded by $K_{\mathbb{R}^+}$, we obtain that $K_{\mathcal{G}} \leq K_{\mathbb{R}^+}$. In the same way, we have $E(u^*, \mathbb{R}^+) \leq E(u, \mathcal{G})$, and hence $\mathcal{E}_{\mathbb{R}^+}(\mu) \leq E(u, \mathcal{G})$, where $\mu := \|u\|_{L^2(\mathcal{G})}^2 = \|u^*\|_{L^2(\mathbb{R}^+)}^2$. By the arbitrariness of u , we obtain the first inequality in (18).

Finally, let $H_{\mu,c}^1$ denote the set of all $u \in H_{\mu}^1(\mathbb{R})$ with compact support. By a density argument, we have

$$\mathcal{E}_{\mathbb{R}}(\mu) = \inf_{u \in H_{\mu,c}^1(\mathbb{R})} E(u, \mathbb{R}) \geq \inf_{u \in H_{\mu}^1(\mathcal{G})} E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu)$$

(the inequality follows from Remark 2.2). This proves the second inequality in (18); the first inequality in (17) is proved in the same way, working with the supremum in (11). \square

After this discussion, we summarize in the next proposition the properties that hold for a generic noncompact graph, without any additional assumption.

Proposition 2.4. *Let \mathcal{G} be a noncompact metric graph.*

- (i) *If $\mu \leq \mu_{\mathcal{G}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) = 0$, and is never attained when $\mu < \mu_{\mathcal{G}}$.*
- (ii) *If $\mu > \mu_{\mathcal{G}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ (possibly $-\infty$).*
- (iii) *If $\mu > \mu_{\mathbb{R}}$, then $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$.*

Proof. When $\mu \leq \mu_{\mathcal{G}}$, (12) shows that $\mathcal{E}_{\mathcal{G}}(\mu) \geq 0$. On the other hand, we infer from (18) that $\mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathbb{R}}(\mu)$, and the latter is zero due to (7), since $\mu \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}$ by (16). Moreover, by (13) we see that $\mathcal{E}_{\mathcal{G}}(\mu)$ is not attained when $\mu < \mu_{\mathcal{G}}$. This proves (i).

By (15), one immediately obtains (ii).

Finally, to prove (iii), observe that $\mathcal{E}_{\mathcal{G}}(\mu) \leq \mathcal{E}_{\mathbb{R}}(\mu) = -\infty$, due to (16) and (7). □

Corollary 2.5. *A necessary condition for the existence of a ground state of mass μ in (4) is that $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$.*

3 Statement of the main results

In this section we state the main results of the paper (Theorems 3.1–3.4), thus providing a precise and formal setting for the four possible cases (a)–(d), that were informally described in the Introduction.

The following theorem covers case (a), represented in Fig. 1.

By a “terminal point” (or “tip”) we mean a point x , in the metric graph \mathcal{G} , that corresponds to a *vertex of degree one* in the underlying (combinatorial) graph. Usually, x is one of the two endpoints of a *bounded* edge attached to the rest of \mathcal{G} only at the other endpoint, as a pendant: the only exception is when \mathcal{G} consists of exactly one unbounded edge (i.e. when $\mathcal{G} = \mathbb{R}^+$), in which case the tip x is the origin of the half-line. We point out that the ∞ -point of any half-line of \mathcal{G} (though being a vertex of degree one in the underlying combinatorial graph) is *not* a terminal point, since it is not a point of \mathcal{G} (as a metric graph).

Theorem 3.1 (graphs with a tip). *Let \mathcal{G} be a noncompact metric graph having at least one terminal point (a tip). Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$. When $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$. When $\mu = \mu_{\mathbb{R}^+}$, $\mathcal{E}_{\mathcal{G}}(\mu) = 0$ and it is attained if and only if \mathcal{G} is isometric to a half-line.*

The next theorem covers case (b), when \mathcal{G} admits a cycle covering (see Fig. 2).

Here and throughout, by “cycle” we mean either a *loop* (a homeomorphic image of S^1) or, by extension, an unbounded path that joins two (necessarily distinct) ∞ -points of \mathcal{G} . In the former case the cycle corresponds to a closed path in the underlying combinatorial graph (i.e. it is a “cycle” in the usual sense of graph theory) whereas, in the latter case, it does not (the two notions would essentially coincide, however, if properly reformulated for the one-point compactification of \mathcal{G}). Alternatively, in the underlying combinatorial graph, one might identify all the ∞ -points of \mathcal{G} into a unique, special vertex (of degree equal to the number of half-lines of \mathcal{G}): in this way, also cycles of the second type would be usual cycles in the graph-theoretic sense.

The existence of a cycle covering is equivalent (see [6, 7]) to a property of \mathcal{G} , called “assumption (H)”, first identified in [6] as a topological obstruction to the existence of ground states in the subcritical cases. In particular, this assumption is incompatible with the presence of tips and forces the graph to have at least two half-lines.

Theorem 3.2 (graphs with a cycle covering). *Let \mathcal{G} be a noncompact metric graph that admits a cycle covering. Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$. The infimum $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained if and only if $\mu = \mu_{\mathbb{R}}$ and \mathcal{G} is \mathbb{R} or a “tower of bubbles” (one of the special graphs described in Example 2.4 of [6]).*

A different behaviour occurs if one considers graphs without terminal points and with one half-line only: the critical mass turns out to coincide with $\mu_{\mathbb{R}^+}$, but the ground state

energy level remains finite if the mass does not exceed $\mu_{\mathbb{R}}$. Furthermore, in the interval $(\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}]$ a ground state exists and it has negative energy.

Theorem 3.3 (graphs with one half-line). *Let \mathcal{G} be a noncompact metric graph having exactly one half-line and no terminal point. Then $\mu_{\mathcal{G}} = \mu_{\mathbb{R}+}$. The infimum $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained if and only if $\mu \in (\mu_{\mathbb{R}+}, \mu_{\mathbb{R}}]$.*

The last theorem deals with the remaining cases, under the additional hypothesis $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$. Notice that, for such graphs, the interval of masses where a ground state exists, is closed.

Theorem 3.4. *Let \mathcal{G} be a noncompact metric graph with no terminal point, having at least two half-lines and admitting no cycle covering. If $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$, then for every $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$, the infimum $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained.*

The class of graphs satisfying the assumptions of Theorem 3.4 is not empty, since it contains, for instance, the “signpost” graph \mathcal{G} of Fig. 4. To see this we take a soliton ϕ (necessarily of mass $\mu_{\mathbb{R}}$) as defined in (8) and we apply the procedure detailed in Sec. 3 of [7]. This produces a function $u \in H_{\mu_{\mathbb{R}}}^1(\mathcal{G})$ such that $E(u, \mathcal{G}) < E(\phi, \mathbb{R}) = 0$. Therefore, $\mathcal{E}_{\mathcal{G}}(\mu_{\mathbb{R}}) < 0$ and, by Proposition 2.4, we conclude that $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$.

Remark 3.5. The last four theorems also provide an answer to the question of existence of extremal functions for the Gagliardo-Nirenberg inequality (10). The key observation is that, for any noncompact graph \mathcal{G} , the following two conditions are in fact equivalent:

- (i) there exists $u \in H^1(\mathcal{G})$, $u \not\equiv 0$, achieving equality in (10);
- (ii) the infimum $\mathcal{E}_{\mathcal{G}}(\mu_{\mathcal{G}})$ is attained by a ground state of mass $\mu_{\mathcal{G}}$.

Indeed, by homogeneity, a function extremal for (10) can be supposed to have mass $\mu_{\mathcal{G}}$: on the other hand, since $K_{\mathcal{G}} \mu_{\mathcal{G}}^2 = 3$, optimality in (10) combined with $\|u\|_{L^2}^2 = \mu_{\mathcal{G}}$ is equivalent to $E(u, \mathcal{G}) = 0$, i.e. to u being a ground state, by Proposition 2.4.

We end this section with the short proofs of the first two theorems. The last two theorems are, on the contrary, much more involved, and to their proofs are devoted the last sections of the paper.

Proof of Theorem 3.1. Let $\mu > \mu_{\mathbb{R}+}$. For every $\varepsilon > 0$, there exists $u \in H_{\mu}^1(\mathbb{R}^+)$ with compact support such that

$$(19) \quad \frac{\|u\|_{L^6(\mathbb{R}^+)}^6}{\mu^2 \cdot \|u'\|_{L^2(\mathbb{R}^+)}^2} \geq K_{\mathbb{R}^+} - \varepsilon.$$

Replacing u by $u_{\lambda}(x) = \sqrt{\lambda}u(\lambda x)$ with λ large, we can assume that the support of u_{λ} is contained in an interval shorter than a terminal edge of \mathcal{G} . Then the function u_{λ} can be seen as an element of $H_{\mu}^1(\mathcal{G})$ (place the support of u on a terminal edge of \mathcal{G} , and set $u_{\lambda} \equiv 0$ elsewhere on \mathcal{G}). Consequently, the quotient in the preceding inequality does not exceed $K_{\mathcal{G}}$. Thus, for every $\varepsilon > 0$, $K_{\mathcal{G}} \geq K_{\mathbb{R}^+} - \varepsilon$ and, recalling (17), we obtain $K_{\mathcal{G}} = K_{\mathbb{R}^+}$, that is, $\mu_{\mathcal{G}} = \mu_{\mathbb{R}+}$.

To prove that $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$, notice that since $K_{\mathcal{G}} = K_{\mathbb{R}^+}$, (19) readily implies, for ε small, that $E(u_{\lambda}, \mathcal{G}) < 0$ (as in (14)). Therefore

$$E(u_{\lambda}, \mathcal{G}) = E(u_{\lambda}, \mathbb{R}^+) = \lambda^2 E(u, \mathbb{R}^+) \rightarrow -\infty$$

as $\lambda \rightarrow +\infty$. Thus, $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$.

Finally, let $\mu = \mu_{\mathbb{R}^+}$. If $\mathcal{G} = \mathbb{R}^+$, plainly $\mathcal{E}_{\mathbb{R}^+}(\mu_{\mathbb{R}^+})$ is attained by the half-solitons. Conversely, assume that there exists $u \in H_{\mu_{\mathbb{R}^+}}^1(\mathcal{G})$ such that $E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu_{\mathbb{R}^+}) = 0$. Its decreasing rearrangement u^* is in $H_{\mu_{\mathbb{R}^+}}^1(\mathbb{R}^+)$ and satisfies $E(u^*, \mathbb{R}^+) \leq E(u, \mathcal{G}) = 0$. Since $\mathcal{E}_{\mathbb{R}^+}(\mu_{\mathbb{R}^+}) = 0$, the function u^* must be a half-soliton. From $E(u, \mathcal{G}) = E(u^*, \mathbb{R}^+)$ we deduce (via Proposition 3.1 of [6]) that almost every point in the range of u has exactly one preimage. As u and u^* are equimeasurable, it follows that u is injective, which means that u is supported on a subset of a half-line. But as $E(u, \mathcal{G}) = 0$, also u must be a half-soliton and this is possible only if there are no vertices other than the ends of the half-line, i.e., \mathcal{G} is (isometric to) \mathbb{R}^+ . \square

Proof of Theorem 3.2. Take any nonnegative $v \in H^1(\mathcal{G}) \setminus \{0\}$, and let $\widehat{v} \in H^1(\mathbb{R})$ denote its symmetric rearrangement on \mathbb{R} . Since \mathcal{G} admits a cycle covering (namely, it satisfies assumption (H)), we have (see Proposition 3.1 in [6])

$$\|\widehat{v}'\|_{L^2(\mathbb{R})} \leq \|v'\|_{L^2(\mathcal{G})}$$

and therefore

$$\frac{\|v\|_{L^6(\mathcal{G})}^6}{\|v\|_{L^2(\mathcal{G})}^4 \cdot \|v'\|_{L^2(\mathcal{G})}^2} \leq \frac{\|\widehat{v}\|_{L^6(\mathbb{R})}^6}{\|\widehat{v}\|_{L^2(\mathbb{R})}^4 \cdot \|\widehat{v}'\|_{L^2(\mathbb{R})}^2} \leq K_{\mathbb{R}},$$

showing that $K_{\mathcal{G}} \leq K_{\mathbb{R}}$. By (17) we obtain $K_{\mathcal{G}} = K_{\mathbb{R}}$, namely $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$.

If $\mu \neq \mu_{\mathbb{R}} = \mu_{\mathcal{G}}$, Proposition 2.4 shows that $\mathcal{E}_{\mathcal{G}}(\mu)$ is not attained.

If $\mu = \mu_{\mathbb{R}}$, a soliton can be placed on \mathbb{R} or on any tower of bubbles (see [6]), showing that those graphs carry a ground state. Conversely, assume that there exists $u \in H_{\mu_{\mathbb{R}}}^1(\mathcal{G})$ such that $E(u, \mathcal{G}) = \mathcal{E}_{\mathcal{G}}(\mu_{\mathbb{R}}) = 0$. Since \mathcal{G} satisfies assumption (H), the symmetric rearrangement \widehat{u} is in $H_{\mu_{\mathbb{R}}}^1(\mathbb{R})$ and satisfies $E(\widehat{u}, \mathbb{R}) \leq E(u, \mathcal{G}) = 0$. Since $\mathcal{E}_{\mathbb{R}}(\mu_{\mathbb{R}}) = 0$, the function \widehat{u} is a soliton. Furthermore, since $E(u, \mathcal{G}) = E(\widehat{u}, \mathbb{R})$, almost every point in the range of u has exactly two preimages. These features show, as in Theorem 2.5 of [6], that \mathcal{G} must be one of the special graphs of [6]. \square

4 A general existence argument

We now present a general existence result for ground states, which is the common core of the proofs of Theorems 3.3 and 3.4, as long as $\mu_{\mathcal{G}} < \mu \leq \mu_{\mathbb{R}}$.

Proposition 4.1. *Let \mathcal{G} be a noncompact metric graph with no terminal point, such that $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$. For every $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$ the infimum $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained.*

The proof is quite involved, and will rely on the following three lemmas.

Lemma 4.2. *Assume \mathcal{G} is noncompact and let $v_n \in H^1(\mathcal{G})$ be a sequence of functions such that $v_n \rightharpoonup 0$ in $H^1(\mathcal{G})$. Then, as $n \rightarrow \infty$,*

$$(20) \quad E(v_n, \mathcal{G}) \geq \frac{1}{2} \left(1 - \frac{\|v_n\|_{L^2(\mathcal{G})}^4}{\mu_{\mathbb{R}}^2} \right) \|v_n'\|_{L^2(\mathcal{G})}^2 + o(1).$$

Proof. Let \mathcal{K} be the compact core of \mathcal{G} (i.e., the compact metric graph obtained from \mathcal{G} by removing the interior of every halfline) and set

$$\varepsilon_n := \max_{x \in \mathcal{K}} |v_n(x)|, \quad w_n(x) = \begin{cases} v_n(x) + \varepsilon_n & \text{if } v_n(x) \leq -\varepsilon_n \\ 0 & \text{if } |v_n(x)| < \varepsilon_n \\ v_n(x) - \varepsilon_n & \text{if } v_n(x) \geq \varepsilon_n \end{cases}$$

Since $v_n \rightharpoonup 0$ in $H^1(\mathcal{G})$, we have $v_n \rightarrow 0$ in $L_{\text{loc}}^\infty(\mathcal{G})$ and hence $\varepsilon_n \rightarrow 0$. Moreover, as $\|v_n - w_n\|_{L^\infty} \leq \varepsilon_n$, from equiboundedness in $L^2(\mathcal{G})$ we have

$$(21) \quad \|v_n - w_n\|_{L^6(\mathcal{G})} \rightarrow 0.$$

Now let \mathcal{H} be an arbitrary halfline of \mathcal{G} . By choosing a coordinate $t \geq 0$ on it, we can identify \mathcal{H} with $[0, +\infty)$ and, since $w_n \equiv 0$ on \mathcal{K} and \mathcal{H} is attached to \mathcal{K} at $t = 0$, we have $w_n(0) = 0$. Setting $w_n(t) \equiv 0$ for $t < 0$, the restriction of w_n to \mathcal{H} can be seen as a function in $H^1(\mathbb{R})$ and, as such, it must obey the following Gagliardo-Nirenberg inequality:

$$\|w_n\|_{L^6(\mathcal{H})}^6 \leq 3 \frac{\|w_n\|_{L^2(\mathcal{H})}^4}{\mu_{\mathbb{R}}^2} \|w_n'\|_{L^2(\mathcal{H})}^2 \leq 3 \frac{\|w_n\|_{L^2(\mathcal{G})}^4}{\mu_{\mathbb{R}}^2} \|w_n'\|_{L^2(\mathcal{H})}^2.$$

Summing over all the halflines of \mathcal{G} , since $w_n \equiv 0$ on \mathcal{K} we obtain

$$\|w_n\|_{L^6(\mathcal{G})}^6 \leq 3 \frac{\|w_n\|_{L^2(\mathcal{G})}^4}{\mu_{\mathbb{R}}^2} \|w_n'\|_{L^2(\mathcal{G})}^2$$

(just as it would be if we had $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$). Since $\|w_n'\|_{L^2(\mathcal{G})}^2 \leq \|v_n'\|_{L^2(\mathcal{G})}^2$ and $\|w_n\|_{L^2(\mathcal{G})}^2 \leq \|v_n\|_{L^2(\mathcal{G})}^2$, from (21) and the previous inequality we obtain,

$$\|v_n\|_{L^6(\mathcal{G})}^6 + o(1) = \|w_n\|_{L^6(\mathcal{G})}^6 \leq 3 \frac{\|v_n\|_{L^2(\mathcal{G})}^4}{\mu_{\mathbb{R}}^2} \|v_n'\|_{L^2(\mathcal{G})}^2,$$

and (20) follows immediately from the definition of $E(v_n, \mathcal{G})$. \square

Lemma 4.3. *Let \mathcal{G} be a noncompact graph, and take $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$. be a nonnegative minimizing sequence for $E(\cdot, \mathcal{G})$ such that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, for some $u \in H^1(\mathcal{G})$.*

If $u \not\equiv 0$, then $u \in H_\mu^1(\mathcal{G})$ and u is a minimizer.

Proof. Passing to a subsequence, we may assume that $u_n(x) \rightarrow u(x)$ a.e. in \mathcal{G} . By a standard use of the Brezis–Lieb Lemma ([13]), as in [7],

$$E(u_n, \mathcal{G}) = E(u_n - u, \mathcal{G}) + E(u, \mathcal{G}) + o(1)$$

as $n \rightarrow \infty$. Since $u_n - u \rightharpoonup 0$ in $H^1(\mathcal{G})$, from Lemma 4.2 applied with $v_n = u_n - u$ we obtain

$$E(u_n - u, \mathcal{G}) \geq \frac{1}{2} \left(1 - \frac{\|u_n - u\|_{L^2(\mathcal{G})}^4}{\mu_{\mathbb{R}}^2} \right) \|u_n' - u'\|_{L^2(\mathcal{G})}^2 + o(1)$$

as $n \rightarrow \infty$. Therefore

$$E(u_n, \mathcal{G}) \geq E(u, \mathcal{G}) + o(1),$$

that is,

$$E(u, \mathcal{G}) \leq \mathcal{E}_{\mathcal{G}}(\mu).$$

Now, by semicontinuity, we have $m := \|u\|_{L^2(\mathcal{G})}^2 \leq \mu$, and $m > 0$ by assumption. If $m < \mu$, then

$$E\left(\sqrt{\frac{\mu}{m}}u, \mathcal{G}\right) = \frac{\mu}{m} \frac{1}{2} \int_{\mathcal{G}} |u'|^2 dx - \left(\frac{\mu}{m}\right)^3 \frac{1}{6} \int_{\mathcal{G}} |u|^6 dx < \frac{\mu}{m} E(u, \mathcal{G}) \leq \mathcal{E}_{\mathcal{G}}(\mu),$$

since $\mu/m > 1$, $\|u\|_{L^6(\mathcal{G})} > 0$ and $\mathcal{E}_{\mathcal{G}}(\mu) \leq 0$. This contradicts the definition of $\mathcal{E}_{\mathcal{G}}(\mu)$. Then it must be $m = \mu$, namely u is the required minimizer. It is also easy to see that the convergence of u_n to u is strong in $H^1(\mathcal{G})$. \square

The next lemma establishes a crucial modification of the Gagliardo–Nirenberg inequality. Its proof is quite long, and is therefore split in a series of steps.

Lemma 4.4 (modified G-N inequality). *Assume \mathcal{G} is noncompact and has no terminal point, and let $u \in H_\mu^1(\mathcal{G})$ for some $\mu \in (0, \mu_\mathbb{R}]$. Then there exists a number $\theta \in [0, \mu]$ such that*

$$(22) \quad \|u\|_{L^6(\mathcal{G})}^6 \leq 3 \left(\frac{\mu - \theta}{\mu_\mathbb{R}} \right)^2 \|u'\|_{L^2(\mathcal{G})}^2 + C\theta^{1/2},$$

where $C > 0$ is a constant that depends only on \mathcal{G} .

Proof. Replacing u with $|u|$, we may assume that $u \geq 0$ and $u \not\equiv 0$.

Step 1. There exist $\ell > 0$ (depending only on \mathcal{G}) and a function $\psi \in H^1((-\infty, \ell])$, such that

1. $\int_{-\infty}^\ell |\psi|^2 dx = \int_{\mathcal{G}} |u|^2 dx = \mu$
2. $\int_{-\infty}^\ell |\psi|^6 dx = \int_{\mathcal{G}} |u|^6 dx$, while $\int_{-\infty}^\ell |\psi'|^2 dx \leq \int_{\mathcal{G}} |u'|^2 dx$;
3. ψ is nonnegative on $(-\infty, \ell]$, and nonincreasing on $[0, \ell]$;

Proof of step 1. If \mathcal{G} has at least a loop (a homeomorphic image of \mathcal{S}^1), let 2ℓ denote the length of the shortest loop; otherwise, let $\ell := 1$. Let us denote by M the maximum of u on \mathcal{G} , and by x_0 a point of \mathcal{G} such that $u(x_0) = M$: since \mathcal{G} is noncompact and connected, there is a path Γ in \mathcal{G} that joins an ∞ -point of \mathcal{G} to x_0 .

Since x_0 is not a terminal point of \mathcal{G} , the path Γ can be prolonged beyond x_0 , to a longer path that crosses x_0 . Two cases are possible: (i) Γ can be prolonged beyond x_0 for a length ℓ , or (ii) a self intersection occurs (i.e. a loop is created) before an extra length of ℓ has been traveled.

In case (i), let γ denote the new path (of length ℓ , starting at x_0) used to prolong Γ (note: γ shares with Γ only the point x_0). We then define the function $\psi : [0, \ell] \rightarrow \mathbb{R}$ as the decreasing rearrangement of u_γ (i.e. u restricted to γ), so that

$$\psi(0) = M, \quad \int_0^\ell |\psi'|^2 dx \leq \int_\gamma |u'|^2 dx, \quad \int_0^\ell |\psi|^p dx = \int_\gamma |u|^p dx \quad \forall p.$$

Now assume, for a while, that the metric graph $\overline{\mathcal{G} \setminus \gamma}$ (the closure of $\mathcal{G} \setminus \gamma$) is *connected*: in this case, we can consider the function $u^* : [0, +\infty) \rightarrow \mathbb{R}$, defined as the decreasing rearrangement of the restricted function $u_{\overline{\mathcal{G} \setminus \gamma}}$, and observe that, as before,

$$u^*(0) = M, \quad \int_0^\infty |(u^*)'|^2 dx \leq \int_{\mathcal{G} \setminus \gamma} |u'|^2 dx, \quad \int_0^\infty |u^*|^p dx = \int_{\mathcal{G} \setminus \gamma} |u|^p dx \quad \forall p.$$

Then the construction of ψ is easily completed, by letting $\psi(x) = u^*(-x)$ for every $x < 0$.

In general, though, $\overline{\mathcal{G} \setminus \gamma}$ can be disconnected (which would prevent the use of the monotone rearrangement): one of its connected component \mathcal{G}_0 , however, contains the original path Γ and, since $u \geq 0$ and Γ contains a half-line along which u tends to zero, the range of u_Γ (u restricted to Γ) is the interval $[0, M]$, that is, the full range of u . Therefore, by a graph-surgery procedure, every other connected component \mathcal{G}_j ($j > 0$) of $\overline{\mathcal{G} \setminus \gamma}$ can still be “reattached” to Γ (hence to \mathcal{G}_0) as follows: if \mathcal{G}_j was originally attached to γ at a point y_j (not necessarily unique), take $z_j \in \Gamma$ such that $u(z_j) = u(y_j)$, and

attach y_j at z_j (i.e., identify y_j with z_j). By this trick (which preserves the continuity of u , its integral norms and those of u') the restricted function $u_{\overline{\mathcal{G} \setminus \gamma}}$ can now be seen as if defined on a connected graph, and the theory of rearrangements becomes available: then, the construction of ψ can be completed as before.

In case (ii), let $x_1 \in \mathcal{G}$ be the point where the self-intersection occurs, and let γ denote the new-added curve, from x_0 to x_1 . The length of γ is at most ℓ , otherwise we would be in case (i), and hence (since \mathcal{G} has no loop of length smaller than 2ℓ) we see that $x_1 \in \Gamma$ (in other words, the self-intersection occurs along Γ , not along γ , which is a simple arc). For the same reason, the length of the complementary arc $\gamma' \subset \Gamma$, from x_1 to x_0 , is *at least* ℓ . But then, observing that $\Gamma' := (\Gamma \setminus \gamma') \cup \gamma$ is still a path from an ∞ -point of \mathcal{G} to x_0 , the proof can be completed as in case (i), with Γ' and (a suitable portion of) γ' (the latter prolonging the former, for a length of ℓ) now playing the roles of the original Γ and γ .

Step 2. There exist $x_0 \in [\ell/2, \ell)$ and $v \in H^1(\mathbb{R}^+)$ such that, defining

$$(23) \quad \theta := \frac{1}{2} \int_{x_0}^{\ell} \psi^2 dx,$$

there hold:

- i) $v(0) = \psi(0)$;
- ii) $\int_0^{\infty} |v|^2 dx = \int_0^{\ell} |\psi|^2 dx - \theta$;
- iii) $\int_0^{\infty} |v'|^2 dx \leq \int_0^{\ell} |\psi'|^2 dx + C\theta^{1/2}$;
- iv) $\int_0^{\infty} |v|^6 dx \geq \int_0^{\ell} |\psi|^6 dx - C\theta$;

the constant $C > 0$ depends only on \mathcal{G} .

Proof of step 2. Here we shall work with ψ restricted to the interval $[0, \ell]$.

We first prove the existence of $x_0 \in [\ell/2, \ell)$ such that

$$(24) \quad |\psi(x_0)|^4 \leq \frac{64m^{1/2}}{\ell^2} \left(\int_{x_0}^{\ell} |\psi|^2 dx \right)^{3/2}, \quad m := \int_{\ell/2}^{\ell} |\psi|^2 dx.$$

To see this, let $F(x) = \int_x^{\ell} |\psi|^2 dx$. If the inequality in (24) were false for every $x_0 \in [\ell/2, \ell)$, we would have

$$-F'(x) = \psi(x)^2 > \frac{8m^{1/4}}{\ell} \left(\int_x^{\ell} |\psi|^2 dx \right)^{3/4} = \frac{8m^{1/4}}{\ell} F(x)^{3/4}$$

for every $x \in [\ell/2, \ell)$. Therefore,

$$-\left(F(x)^{1/4}\right)' > \frac{2m^{1/4}}{\ell} \quad \forall x \in [\ell/2, \ell)$$

and, since $F(\ell) = 0$, integration over $(\ell/2, \ell)$ yields

$$F(\ell/2)^{1/4} > \frac{2m^{1/4}}{\ell} \cdot \frac{\ell}{2} = m^{1/4},$$

which is clearly a contradiction due to how m and F were defined.

We now take a point x_0 satisfying (24), and define θ as in (23). If $\theta = 0$, then ψ vanishes on $[x_0, \ell]$; in this case we define v by extending ψ to 0 on $[\ell, +\infty)$. If $\theta \neq 0$, we define $v : [0, +\infty) \rightarrow \mathbb{R}$ as

$$v(x) = \begin{cases} \psi(x) & \text{if } 0 \leq x \leq x_0, \\ \psi(x_0)e^{-\lambda(x-x_0)} & \text{if } x > x_0 \end{cases}$$

where

$$\lambda := \frac{|\psi(x_0)|^2}{2\theta}.$$

Clearly $v \in H^1(\mathbb{R}^+)$ and $v(0) = \psi(0)$, so that $i)$ is satisfied. Next,

$$\begin{aligned} \int_0^\infty |v|^2 dx &= \int_0^{x_0} |\psi|^2 dx + |\psi(x_0)|^2 \int_{x_0}^\infty e^{-2\lambda(x-x_0)} dx = \int_0^{x_0} |\psi|^2 dx + \frac{|\psi(x_0)|^2}{2\lambda} \\ (25) \quad &= \int_0^{x_0} |\psi|^2 dx + \theta = \int_0^\ell |\psi|^2 dx - \theta, \end{aligned}$$

and $ii)$ is proved. Similarly,

$$\begin{aligned} \int_0^\infty |v'|^2 dx &= \int_0^{x_0} |\psi'|^2 dx + \lambda^2 |\psi(x_0)|^2 \int_{x_0}^\infty e^{-2\lambda(x-x_0)} dx = \int_0^{x_0} |\psi'|^2 dx + \frac{\lambda |\psi(x_0)|^2}{2} \\ &= \int_0^{x_0} |\psi'|^2 dx + \frac{|\psi(x_0)|^4}{4\theta} \leq \int_0^\ell |\psi'|^2 dx + \frac{32(2m)^{1/2}}{\ell^2} \theta^{1/2} \\ (26) \quad &\leq \int_0^\ell |\psi'|^2 dx + C\theta^{1/2} \end{aligned}$$

since $m \leq \mu_{\mathbb{R}} = \pi\sqrt{3}/2$ by (7), while ℓ depends only on \mathcal{G} . This proves $iii)$. Finally,

$$\begin{aligned} \int_0^\infty |v|^6 dx &= \int_0^{x_0} |\psi|^6 dx + |\psi(x_0)|^6 \int_{x_0}^\infty e^{-6\lambda(x-x_0)} dx = \int_0^{x_0} |\psi|^6 dx + \frac{|\psi(x_0)|^6}{6\lambda} \\ (27) \quad &\geq \int_0^\ell |\psi|^6 dx - \int_{x_0}^\ell |\psi|^6 dx. \end{aligned}$$

Since ψ is decreasing (on $[0, \ell]$) and $x_0 \geq \ell/2$, we have

$$\int_{x_0}^\ell |\psi|^6 dx \leq |\psi(x_0)|^4 \int_{x_0}^\ell |\psi|^2 dx \leq 2\theta |\psi(\ell/2)|^4$$

and

$$|\psi(\ell/2)|^2 \leq \frac{2}{\ell} \int_0^{\ell/2} |\psi|^2 dx < \frac{2\mu}{\ell} \leq C$$

as above, which plugged into the previous inequality gives

$$\int_{x_0}^\ell |\psi|^6 dx \leq C\theta.$$

Therefore (27) reads

$$\int_0^\infty |v|^6 dx \geq \int_0^\ell |\psi|^6 dx - C\theta,$$

and this concludes the proof.

Step 3. Combining ψ and v , we now define

$$w(x) = \begin{cases} \psi(x) & \text{if } x \leq 0, \\ v(x) & \text{if } x > 0. \end{cases}$$

Clearly $w \in H^1(\mathbb{R})$ and, by the properties of ψ and v ,

$$\int_{\mathbb{R}} |w|^2 dx = \int_{-\infty}^0 |\psi|^2 dx + \int_0^\infty |v|^2 dx = \int_{-\infty}^\ell |\psi|^2 dx - \theta = \int_{\mathcal{G}} |u|^2 dx - \theta = \mu - \theta.$$

By the Gagliardo–Nirenberg inequality (10),

$$\begin{aligned} \|w\|_{L^6(\mathbb{R})}^6 &\leq K_{\mathbb{R}} (\mu - \theta)^2 \|w'\|_{L^2(\mathbb{R})}^2 = K_{\mathbb{R}} \mu_{\mathbb{R}}^2 \left(\frac{\mu - \theta}{\mu_{\mathbb{R}}} \right)^2 \|w'\|_{L^2(\mathbb{R})}^2 \\ (28) \quad &= 3 \left(\frac{\mu - \theta}{\mu_{\mathbb{R}}} \right)^2 \|w'\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Now, still from the properties of ψ and v ,

$$\begin{aligned} \|w\|_{L^6(\mathbb{R})}^6 &= \int_{-\infty}^0 |\psi|^6 dx + \int_0^\infty |v|^6 dx \\ (29) \quad &\geq \int_{-\infty}^0 |\psi|^6 dx + \int_0^\ell |\psi|^6 dx - C\theta = \|u\|_{L^6(\mathcal{G})}^6 - C\theta \end{aligned}$$

and

$$\begin{aligned} \|w'\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^0 |\psi'|^2 dx + \int_0^\infty |v'|^2 dx \\ (30) \quad &\leq \int_{-\infty}^0 |\psi'|^2 dx + \int_0^\ell |\psi'|^2 dx + C\theta^{1/2} \leq \|u'\|_{L^2(\mathcal{G})}^2 + C\theta^{1/2}. \end{aligned}$$

Inserting (29) and (30) in (28) we obtain

$$\|u\|_{L^6(\mathcal{G})}^6 - C\theta \leq 3 \left(\frac{\mu - \theta}{\mu_{\mathbb{R}}} \right)^2 \left(\|u'\|_{L^2(\mathcal{G})}^2 + C\theta^{1/2} \right).$$

Rearranging terms and observing that $\theta \leq \theta^{1/2} \mu_{\mathbb{R}}^{1/2} = C\theta^{1/2}$, one obtains (22). □

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Fix a mass $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$. We know from Proposition 2.4 that $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ (possibly $-\infty$, until proven otherwise), thus we only consider functions $u \in H_{\mu}^1(\mathcal{G})$ such that $E(u, \mathcal{G}) \leq -\alpha$ for some fixed $\alpha > 0$ that depends on μ . Let u be any of these functions: since $\mu \leq \mu_{\mathbb{R}}$, Lemma 4.4 applies, and (22) yields

$$\|u\|_{L^6(\mathcal{G})}^6 \leq 3 \left(1 - \frac{\theta_u}{\mu_{\mathbb{R}}} \right)^2 \|u'\|_{L^2(\mathcal{G})}^2 + C\theta_u^{1/2}$$

for some $\theta_u \in (0, \mu)$. We have denoted by θ_u the constant θ appearing in (22) to stress its dependence on u . Rearranging terms we obtain

$$3 \frac{\theta_u}{\mu_{\mathbb{R}}} \left(2 - \frac{\theta_u}{\mu_{\mathbb{R}}} \right) \|u'\|_{L^2(\mathcal{G})}^2 - C\theta_u^{1/2} \leq 6E(u, \mathcal{G}) \leq -6\alpha,$$

and (since $\theta_u < \mu \leq \mu_{\mathbb{R}}$) this shows that θ_u is bounded away from zero (in terms of C and α , uniformly with respect to u). Once this is established, the same inequality also shows that $\|u'\|_{L^2(\mathcal{G})}$ is bounded from above, in terms of C and α . We have thus proved that whenever $E(u, \mathcal{G}) \leq \alpha < 0$, the L^2 norm of u' on \mathcal{G} is uniformly bounded (the bounding constant depends only on \mathcal{G} and μ , via C and α). By the Gagliardo–Nirenberg inequality the same holds for $\|u\|_{L^p(\mathcal{G})}$, for every $p \in [2, +\infty]$. In particular, $\mathcal{E}_{\mathcal{G}}(\mu)$ is finite.

Finally, we show that $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained.

Let $u_n \in H_{\mu}^1(\mathcal{G})$ be a minimizing sequence for $E(\cdot, \mathcal{G})$. By the preceding argument we can assume that $\|u'_n\|_{L^2(\mathcal{G})}$, $\|u_n\|_{L^6(\mathcal{G})}$ and $\|u_n\|_{L^\infty(\mathcal{G})}$ are bounded independently of n . Up to subsequences, $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, as well as $u_n \rightarrow u$ in $L_{\text{loc}}^q(\mathcal{G})$ for every $q \in [2, \infty]$.

If $u \equiv 0$, by Lemma 4.2,

$$E(u_n, \mathcal{G}) \geq \frac{1}{2} \left(1 - \frac{\mu^2}{\mu_{\mathbb{R}}^2} \right) \|u'_n\|_{L^2(\mathcal{G})}^2 + o(1) \geq o(1),$$

since $\mu \leq \mu_{\mathbb{R}}$. This implies $\mathcal{E}_{\mathcal{G}}(\mu) \geq 0$, which is false. Thus, the weak limit u is not identically zero and, by Lemma 4.3, is the required minimizer.

5 Proof of the main results

In this section, building on Proposition 4.1, we present the proofs of Theorems 3.3 and 3.4.

Proof of Theorem 3.3. Let ϕ be the function defined by (9), thought of as a half-soliton on \mathbb{R}^+ and notice that $\phi(0) = 1$. Identify \mathcal{H} , the unique half-line of \mathcal{G} , with the interval $[0, +\infty)$ and for $\varepsilon > 0$ set

$$u_\varepsilon(x) = \begin{cases} \sqrt{\varepsilon}\phi(\varepsilon x) & \text{if } x \in \mathcal{H} \text{ (i.e. if } x \in [0, +\infty)) \\ \sqrt{\varepsilon} & \text{elsewhere on } \mathcal{G}. \end{cases}$$

Observe that $\mathcal{G} \setminus \mathcal{H} \neq \emptyset$, because \mathcal{G} has no terminal point. Since \mathcal{G} is connected and \mathcal{H} is attached to $\mathcal{G} \setminus \mathcal{H}$ at $x = 0$, we see that $u_\varepsilon \in H^1(\mathcal{G})$ and hence

$$K_{\mathcal{G}} \geq \frac{\|u_\varepsilon\|_{L^6(\mathcal{G})}^6}{\|u_\varepsilon\|_{L^2(\mathcal{G})}^4 \|u'_\varepsilon\|_{L^2(\mathcal{G})}^2} = \frac{\|\phi\|_{L^6(\mathbb{R}^+)}^6 + \ell\varepsilon^3}{(\|\phi\|_{L^2(\mathbb{R}^+)}^2 + \ell\varepsilon)^2 \|\phi'\|_{L^2(\mathbb{R}^+)}^2},$$

where ℓ is the measure (i.e. the total length) of $\mathcal{G} \setminus \mathcal{H}$. Since the last quotient tends to $K_{\mathbb{R}^+}$ as $\varepsilon \rightarrow 0$, we have $K_{\mathcal{G}} \geq K_{\mathbb{R}^+}$ and, in fact, equality occurs, by (17). Thus $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$.

Now the fact that $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained when $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$ is the content of Proposition 4.1.

Finally, since $\mu_{\mathbb{R}^+} = \mu_{\mathcal{G}}$, $\mathcal{E}_{\mathcal{G}}(\mu_{\mathbb{R}^+}) = 0$. If $u \in H_{\mu_{\mathbb{R}^+}}^1(\mathcal{G})$ is such that $E(u, \mathcal{G}) = 0$, then its decreasing rearrangement u^* on \mathbb{R}^+ satisfies $E(u^*, \mathbb{R}^+) \leq 0$. Then, exactly as in the last part of the proof of Theorem 3.1, this implies that \mathcal{G} is (isometric to) \mathbb{R}^+ . But this is impossible, since \mathcal{G} has no terminal point. \square

To complete the proof of Theorem 3.4 we need the following lemma.

Lemma 5.1 (Bridge doubling). *Assume \mathcal{G} is noncompact, and let \mathcal{B} denote the union of all those edges that do not belong to any cycle. Then, for every $u \in H_\mu^1(\mathcal{G})$,*

$$(31) \quad \int_{\mathcal{G}} |u|^6 dx + 3 \int_{\mathcal{B}} |u|^6 dx \leq 3 \left(\frac{\mu + 3\mu_{\mathcal{B}}}{\mu_{\mathbb{R}}} \right)^2 \int_{\mathcal{G}} |u'|^2 dx,$$

where

$$(32) \quad \mu_{\mathcal{B}} := \int_{\mathcal{B}} |u|^2 dx.$$

Proof. If $\mathcal{B} = \emptyset$, then \mathcal{G} admits a cycle covering and, by Theorem 3.2, $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$: then $\mu_{\mathcal{B}} = 0$ and (31) reduces to the Gagliardo-Nirenberg inequality. So we may assume $\mathcal{B} \neq \emptyset$: the idea behind the proof is that (31) is still the Gagliardo-Nirenberg inequality, but for a modified function \tilde{u} on a modified graph $\tilde{\mathcal{G}}$ where we force a cycle covering.

We construct $\tilde{\mathcal{G}}$ from \mathcal{G} , as follows: for every edge $e \in \mathcal{B}$, we stretch e by a factor 2, and we *duplicate* the resulting edge, so that, if the original e had length ℓ , there are now *two* edges, each of length 2ℓ , joining the same two vertices (or emanating from the same vertex, if e is a half-line and $\ell = +\infty$). These two new edges now form a cycle, so that the resulting graph $\tilde{\mathcal{G}}$ admits a cycle covering.

Given $u \in H^1(\mathcal{G})$, we construct $\tilde{u} \in H^1(\tilde{\mathcal{G}})$ as follows. First, we let $\tilde{u} \equiv u$ on $\mathcal{G} \setminus \mathcal{B}$. Then, for every edge $e \in \mathcal{B}$, we duplicate u (stretched horizontally by a factor 2) on each of the two copies of the stretched edge $2e$ of $\tilde{\mathcal{G}}$. Choosing a coordinate $x \in [0, \ell]$ on every $e \in \mathcal{B}$ of length ℓ , this amounts to replacing $u(x)$ over $[0, \ell]$ with *two copies* of $u(x/2)$, over $[0, 2\ell]$ (these intervals are to be replaced with $[0, +\infty)$ when e is a half-line). It is then clear that for every $p \geq 1$,

$$(33) \quad \int_{\tilde{\mathcal{G}}} |\tilde{u}|^p dx = \int_{\mathcal{G} \setminus \mathcal{B}} |u|^p dx + 4 \int_{\mathcal{B}} |u|^p dx = \int_{\mathcal{G}} |u|^p dx + 3 \int_{\mathcal{B}} |u|^p dx$$

(in particular, when $p = 2$ and when $p = 6$), and, similarly,

$$(34) \quad \int_{\tilde{\mathcal{G}}} |\tilde{u}'|^2 dx = \int_{\mathcal{G} \setminus \mathcal{B}} |u'|^2 dx + \int_{\mathcal{B}} |u'|^2 dx = \int_{\mathcal{G}} |u'|^2 dx.$$

On the other hand, since $\tilde{\mathcal{G}}$ is covered by cycles, Theorem 3.2 gives $\mu_{\tilde{\mathcal{G}}} = \mu_{\mathbb{R}}$, so that \tilde{u} is subject to a Gagliardo-Nirenberg inequality that can be written

$$\int_{\tilde{\mathcal{G}}} |\tilde{u}|^6 dx \leq 3 \left(\frac{\int_{\tilde{\mathcal{G}}} |\tilde{u}|^2 dx}{\mu_{\mathbb{R}}} \right)^2 \int_{\tilde{\mathcal{G}}} |\tilde{u}'|^2 dx,$$

and (31) follows immediately using (32), (33) and (34). \square

Proof of Theorem 3.4. We recall that $\mu_{\mathcal{G}} < \mu_{\mathbb{R}}$ by assumption. When $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$, the fact that $\mathcal{E}_{\mathcal{G}}(\mu)$ is attained follows from Proposition 4.1.

On the other hand, the proof of Proposition 4.1 cannot be adapted to the case where $\mu = \mu_{\mathcal{G}}$, because it is based on the inequality $\mathcal{E}_{\mathcal{G}}(\mu) < 0$, that now is replaced by $\mathcal{E}_{\mathcal{G}}(\mu_{\mathcal{G}}) = 0$. This is not a weakness of the proof: now, indeed, any sequence $u_n \in H_\mu^1(\mathcal{G})$ such that $\|u'_n\|_{L^2} \rightarrow 0$ is, by virtue of (10), a minimizing sequence, so that minimizing sequences are in general noncompact. To get compactness, one has to carefully select the minimizing sequence, as follows.

Let $u_n \in H^1_{\mu_{\mathcal{G}}}(\mathcal{G})$ be a maximizing sequence for the Gagliardo–Nirenberg inequality, namely a sequence such that

$$(35) \quad \frac{\|u_n\|_{L^6(\mathcal{G})}^6}{\|u'_n\|_{L^2(\mathcal{G})}^2} \rightarrow K_{\mathcal{G}} \mu_{\mathcal{G}}^2 = 3$$

as $n \rightarrow \infty$. Applying inequality (22) to u_n keeping in mind that $\mu = \mu_{\mathcal{G}}$ and $\theta \leq \mu \leq C$ we can get rid of θ and write (for some other C)

$$(36) \quad \|u_n\|_{L^6(\mathcal{G})}^6 \leq 3 \frac{\mu_{\mathcal{G}}^2}{\mu_{\mathbb{R}}^2} \|u'_n\|_{L^2(\mathcal{G})}^2 + C.$$

We first notice that $\|u'_n\|_{L^2(\mathcal{G})}$ (and hence also $\|u_n\|_{L^6(\mathcal{G})}^6$) must be bounded. Indeed, if this is not the case, along a subsequence,

$$\lim_n \frac{\|u_n\|_{L^6(\mathcal{G})}^6}{\|u'_n\|_{L^2(\mathcal{G})}^2} \leq \lim_n \left(3 \frac{\mu_{\mathcal{G}}^2}{\mu_{\mathbb{R}}^2} + \frac{C}{\|u'_n\|_{L^2(\mathcal{G})}^2} \right) = 3 \frac{\mu_{\mathcal{G}}^2}{\mu_{\mathbb{R}}^2} < 3,$$

by (36), contradicting (35).

Now since \mathcal{G} has at least two half-lines, by (31) we have

$$\int_{\mathcal{G}} |u_n|^6 dx + 3 \int_{\mathcal{B}} |u_n|^6 dx \leq 3 \left(\frac{\mu_{\mathcal{G}} + 3\mu_n}{\mu_{\mathbb{R}}} \right)^2 \int_{\mathcal{G}} |u'_n|^2 dx, \quad \mu_n := \int_{\mathcal{B}} |u_n|^2 dx,$$

where \mathcal{B} is defined as in Lemma 5.1. This allows us to show that $\|u_n\|_{L^6(\mathcal{G})}$ and $\|u'_n\|_{L^2(\mathcal{G})}$ are bounded away from zero. If (for some subsequence) the two norms tend to zero, then clearly also $\mu_n \rightarrow 0$. Dividing the preceding inequality by $\|u'_n\|_{L^2(\mathcal{G})}^2$ we obtain

$$\frac{\|u_n\|_{L^6(\mathcal{G})}^6}{\|u'_n\|_{L^2(\mathcal{G})}^2} \leq 3 \left(\frac{\mu_{\mathcal{G}} + 3\mu_n}{\mu_{\mathbb{R}}} \right)^2 = 3 \frac{\mu_{\mathcal{G}}^2}{\mu_{\mathbb{R}}^2} + o(1),$$

and then

$$\lim_n \frac{\|u_n\|_{L^6(\mathcal{G})}^6}{\|u'_n\|_{L^2(\mathcal{G})}^2} < 3,$$

contradicting again (35).

Finally, since u_n is bounded in $H^1(\mathcal{G})$, writing

$$6E(u_n, \mathcal{G}) = 3\|u'_n\|_{L^2(\mathcal{G})}^2 - \|u_n\|_{L^6(\mathcal{G})}^6 = \|u'_n\|_{L^2(\mathcal{G})}^2 \left(3 - \frac{\|u_n\|_{L^6(\mathcal{G})}^6}{\|u'_n\|_{L^2(\mathcal{G})}^2} \right)$$

we see by (35) that $E(u_n, \mathcal{G}) \rightarrow 0 = \mathcal{E}_{\mathcal{G}}(\mu_{\mathcal{G}})$. We have thus constructed a minimizing sequence for $E(\cdot, \mathcal{G})$ which is bounded and uniformly away from zero.

We may then assume that $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, as well as the other usual convergence properties. If $u \equiv 0$, by Lemma 4.2,

$$E(u_n, \mathcal{G}) \geq \frac{1}{2} \left(1 - \frac{\mu_{\mathcal{G}}^2}{\mu_{\mathbb{R}}^2} \right) \|u'_n\|_{L^2(\mathcal{G})}^2 + o(1).$$

Since $E(u_n, \mathcal{G}) \rightarrow 0$, this contradicts the construction of the sequence u_n . Therefore the weak limit u does not vanish identically, and by Lemma 4.3, it is the required minimizer. \square

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